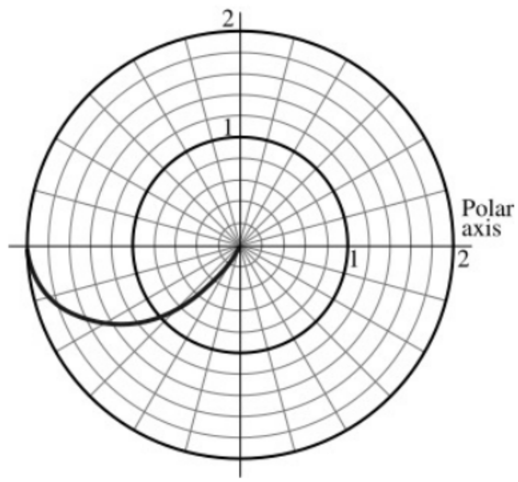
# Spring 2025 Laird Homework 8 Solutions

## Question 1

The figure shows a portion of the graph of the polar function $r = 2 - 4\cos\theta$ in the polar coordinate system for $a \le \theta \le b$.




### Part A

$a = 0$, $b = \frac{\pi}{3}$

Let's look at some key points on the curve.

- When $\theta = 0$: $r = 2 - 4\cos(0) = 2 - (4)(1) = -2$
- When $\theta = \frac{\pi}{3}$: $r = 2 - 4\cos(\frac{\pi}{3}) = 2 - (4)(\frac{1}{2}) = 2 - 2 = 0$
- When $\theta = \frac{\pi}{2}$: $r = 2 - 4\cos(\frac{\pi}{2}) = 2 - (4)(0) = 2$
- When $\theta = \pi$: $r = 2 - 4\cos(\pi) = 2 - (4)(-1) = 2 + 4 = 6$
- When $\theta = 2\pi$: $r = 2 - 4\cos(2\pi) = 2 - (4)(1) = -2$

From $\theta = 0$ to $\theta = \frac{\pi}{3}$, $r$ is negative and increasing, which means the curve is plotted in the third quadrant.

## Question 2

Consider the polar function $r = 3 \cdot (\cos\theta)$.

## Part A

Quadrants I and IV

For the function $r = 3\cos\theta$:

When $0 \leq \theta < \frac{\pi}{2}$: $\cos\theta > 0$, so $r > 0$. The point $(r, \theta)$ is in Quadrant I.

When $\frac{\pi}{2} < \theta < \pi$: $\cos\theta < 0$, so $r < 0$. A negative $r$ means we plot $(|r|, \theta + \pi)$, which puts the point in Quadrant III. However, since we're asking about the original curve, there are no points in Quadrant II.

When $\pi < \theta < \frac{3\pi}{2}$: $\cos\theta < 0$, so $r < 0$. This corresponds to points in Quadrant IV (after converting negative $r$ values).

When $\frac{3\pi}{2} < \theta < 2\pi$: $\cos\theta > 0$, so $r > 0$. The point $(r, \theta)$ is in Quadrant IV.

Because the function is periodic, the curve from $\theta = 0$ to $\theta = 2\pi$ is the same as the curve from $\theta = 2\pi$ to $\theta = 4\pi$, and so on.

Therefore, the curve $r = 3\cos\theta$ contains points only in Quadrants I and IV.

## Part B

Quadrants II and III

For the function $r = -3\cos\theta$:

This is the negative of the previous function. Where $r$ was positive in part (a), it will now be negative, and vice versa.

When $0 \leq \theta < \frac{\pi}{2}$: $\cos\theta > 0$, so now $r < 0$. A negative $r$ means we plot $(|r|, \theta + \pi)$, which puts the point in Quadrant III.

When $\frac{\pi}{2} < \theta < \pi$: $\cos\theta < 0$, so now $r > 0$. The point $(r, \theta)$ is in Quadrant II.

When $\pi < \theta < \frac{3\pi}{2}$: $\cos\theta < 0$, so now $r > 0$. The point $(r, \theta)$ is in Quadrant III.

When $\frac{3\pi}{2} < \theta < 2\pi$: $\cos\theta > 0$, so now $r < 0$. When converted, this puts points in Quadrant II.

Therefore, the curve $r = -3\cos\theta$ contains points only in Quadrants II and III.

## Part C

All four quadrants

For the function $r = 3\cos\theta + 4$:

We need to determine when $r > 0$ and when $r < 0$.

$r = 3\cos\theta + 4 < 0$ when $3\cos\theta < -4$, which means $\cos\theta < -\frac{4}{3}$.

Since $-1 \leq \cos\theta \leq 1$ for all real $\theta$, and $-\frac{4}{3} < -1$, there's no value of $\theta$ for which $r < 0$.

Therefore, $r > 0$ for all values of $\theta$, which means the curve $r = 3\cos\theta + 4$ contains points in all four quadrants.

More generally, the curve $r = a\cos\theta + b$ for some non-zero $b$ will contain points in all four quadrants.

# Question 3

Express the complex number $7 + 7i$ in polar form $r(\cos\theta + i\sin\theta)$.

$7\sqrt{2}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$

For the complex number $7 + 7i$:

$r = \sqrt{a^2 + b^2} = \sqrt{7^2 + 7^2} = \sqrt{98} = 7\sqrt{2}$

$\theta = \arctan(\frac{b}{a}) = \arctan(\frac{7}{7}) = \arctan(1) = \frac{\pi}{4}$

Since the point is in Quadrant I, no adjustment to $\theta$ is needed.

Therefore, $7 + 7i = 7\sqrt{2}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$

# Question 4

Consider the polar function $r = \frac{\theta}{2}$ defined for $\theta \geq 0$.

## Part A

The radius increases linearly with $\theta$ at half the rate of $\theta$. As $\theta$ increases, $r$ increases proportionally but at a slower rate.

For the function $r = \frac{\theta}{2}$, the radius is directly proportional to the angle $\theta$, with a proportionality constant of $\frac{1}{2}$.

As $\theta$ increases, $r$ also increases, but at half the rate of $\theta$. For every increase of 2 radians in $\theta$, the radius $r$ increases by 1 unit.

## Part B

The function forms a spiral that starts at the origin and continuously winds outward. As $\theta$ increases, the distance from the origin increases, creating an unbounded spiral shape.

The function $r = \frac{\theta}{2}$ describes a spiral curve.

When $\theta = 0$, $r = 0$, so the curve starts at the origin.

As $\theta$ increases, $r$ also increases, causing the curve to spiral outward from the origin.

The curve makes a complete turn around the origin when $\theta$ increases by $2\pi$. During this turn, the radius increases by $\pi$ units.

# Question 5

Consider the polar function $r = 2 - \cos(-\theta)$.

## Part A

As $\theta$ changes from $\frac{\pi}{2}$ to $\pi$, the distance between the origin and the point $(f(\theta), \theta)$ increases.

First, note that $\cos(-\theta) = \cos(\theta)$ since cosine is an even function.

So our function simplifies to $r = 2 - \cos(\theta)$.

Now, we analyze how $r$ changes as $\theta$ changes from $\frac{\pi}{2}$ to $\pi$.

When $\theta = \frac{\pi}{2}$: $r = 2 - \cos(\frac{\pi}{2}) = 2 - 0 = 2$

When $\theta = \pi$: $r = 2 - \cos(\pi) = 2 - (-1) = 3$

As $\theta$ increases from $\frac{\pi}{2}$ to $\pi$, $\cos(\theta)$ decreases from 0 to -1.

Therefore, $r = 2 - \cos(\theta)$ increases from 2 to 3.

The distance between the origin and the point $(f(\theta), \theta)$ increases as $\theta$ changes from $\frac{\pi}{2}$ to $\pi$.

## Part B

Minimum value = 2

For the function $r = 2 - \cos(\theta)$ over the interval $[\frac{\pi}{2}, \pi]$:
The minimum value of $r$ will occur when $\cos(\theta)$ is at its maximum, since $\cos(\theta)$ is being subtracted.
In the interval $[\frac{\pi}{2}, \pi]$, the maximum value of $\cos(\theta)$ is $\cos(\frac{\pi}{2}) = 0$.
Therefore, the minimum value of $r$ is $r = 2 - \cos(\frac{\pi}{2}) = 2 - 0 = 2$.

### Part C

Maximum value = 3

For the function $r = 2 - \cos(\theta)$ over the interval $[\frac{\pi}{2}, \pi]$:
The maximum value of $r$ will occur when $\cos(\theta)$ is at its minimum, since $\cos(\theta)$ is being subtracted.
In the interval $[\frac{\pi}{2}, \pi]$, the minimum value of $\cos(\theta)$ is $\cos(\pi) = -1$.
Therefore, the maximum value of $r$ is $r = 2 - \cos(\pi) = 2 - (-1) = 3$.

## Question 6

Consider the polar function $r = 1 + 2\sin\theta$ for $0 \leq \theta \leq 2\pi$.

### Part A

$r$ is decreasing on the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$

For the function $r = 1 + 2\sin\theta$, we need to find where $r$ is decreasing.
$r$ is decreasing when $\sin\theta$ is decreasing.
Therefore, $r$ is decreasing on the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$.

### Part B

The distance is increasing on the intervals $(0, \frac{\pi}{2})$, $(\frac{7\pi}{6}, \frac{3\pi}{2})$, and $(\frac{11\pi}{6}, 2\pi)$

This question asks for intervals where the distance between the point and the origin is increasing.
The distance from the origin is increasing when $r \geq 0$ and increasing **OR** $r \leq 0$ and decreasing.
$r$ is decreasing on the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$, and $r$ is increasing on the intervals $(0, \frac{\pi}{2})$, $(\frac{7\pi}{6}, \frac{3\pi}{2})$, and $(\frac{11\pi}{6}, 2\pi)$.
$r \geq 0$ when $\sin(\theta) \geq -\frac{1}{2}$
$\sin(\theta) \geq -\frac{1}{2}$ when $\theta \in [0, \frac{7\pi}{6}]$ and $\theta \in [\frac{5\pi}{3}, 2\pi]$
Therefore $r \geq 0$ and increasing on the intervals $(0, \frac{\pi}{2})$, $(\frac{7\pi}{6}, \frac{3\pi}{2})$, and $(\frac{11\pi}{6}, 2\pi)$
Additionally, $r \leq 0$ and decreasing on the interval $(\frac{7\pi}{6}, \frac{3\pi}{2})$
Therefore, the distance is increasing on the intervals $(0, \frac{\pi}{2})$, $(\frac{7\pi}{6}, \frac{3\pi}{2})$, and $(\frac{11\pi}{6}, 2\pi)$

## Question 7

Consider the polar function $r = -1 + \sin\theta$

### Part A

Minimum value = -1

For the function $r = -1 + \sin\theta$ over the interval $(0, \frac{\pi}{2})$:
The minimum value of $r$ will occur when $\sin\theta$ is at its minimum in this interval.
In the interval $(0, \frac{\pi}{2})$, $\sin\theta$ increases from $\sin(0) = 0$ to $\sin(\frac{\pi}{2}) = 1$.
The minimum value of $\sin\theta$ in this interval is $\sin(0) = 0$.
Therefore, the minimum value of $r$ is $r = -1 + \sin(0) = -1 + 0 = -1$.

### Part B

The graph lies above or on the $x$-axis in the intervals $[\pi, 2\pi]$

A point in polar coordinates $(r, \theta)$ lies above or on the $x$-axis when its $y$-coordinate is non-negative.
This can occur when $\theta$ is in Quadrants I or II, and $r$ is non-negative.
It can also occur when $\theta$ is in Quadrants III or IV, and $r$ is non-positive.
For our function $r = -1 + \sin\theta$, $r$ is always non-positive, because the maximum value of $\sin(\theta)$ is 1.
Therefore, the graph lies above or on the x-axis when $\theta$ is in Quadrants III or IV.
Our solution is $[\pi, 2\pi]$.

## Question 8

Consider the polar function $r = 2\sin(2\theta)$ for $0 \leq \theta \leq \pi$.

### Part A

$r$ has extrema at $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$

For the function $r = 2\sin(2\theta)$, we need to find where $r$ has extrema in the interval $[0, \pi]$.
$r$ has extrema when $\frac{dr}{d\theta} = 0$.
$\frac{dr}{d\theta} = 4\cos(2\theta)$
Setting this to zero: $4\cos(2\theta) = 0$ $\cos(2\theta) = 0$
This occurs when $2\theta = \frac{\pi}{2} + n\pi$ for integer $n$.
Solving for $\theta$ in the interval $[0, \pi]$: $\theta = \frac{\pi}{4} + \frac{n\pi}{2}$
For $n = 0$: $\theta = \frac{\pi}{4}$ For $n = 1$: $\theta = \frac{3\pi}{4}$
Therefore, $r$ has extrema at $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ in the interval $[0, \pi]$.