Spring 2025 Laird Homework 5 Solutions

Question 1

Part A

$\sin^{-1}(1/2) = \frac{\pi}{6}$

We need to find the angle θ such that $\sin(\theta) = \frac{1}{2}$.

From the unit circle, we know that $\sin(\frac{\pi}{6}) = \frac{1}{2}$. Since $\frac{\pi}{6}$ is in the first quadrant and $\sin^{-1}$ returns values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we have $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$.

Part B

$\cos^{-1}(1/2) = \frac{\pi}{3}$

We need to find the angle θ such that $\cos(\theta) = \frac{1}{2}$.

From the unit circle, we know that $\cos(\frac{\pi}{3}) = \frac{1}{2}$. Since $\frac{\pi}{3}$ is in the first quadrant and $\cos^{-1}$ returns values in $[0, \pi]$, we have $\cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$.

Part C

$\tan^{-1}(1) = \frac{\pi}{4}$

We need to find the angle θ such that $\tan(\theta) = 1$.

From the unit circle, we know that $\tan(\frac{\pi}{4}) = 1$. Since $\frac{\pi}{4}$ is in the first quadrant and $\tan^{-1}$ returns values in $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have $\tan^{-1}(1) = \frac{\pi}{4}$.

Part D

$\sin^{-1}(-1/2) = -\frac{\pi}{6}$

We need to find the angle θ such that $\sin(\theta) = -\frac{1}{2}$.

From the unit circle, we know that $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$. Since $-\frac{\pi}{6}$ is in the fourth quadrant and $\sin^{-1}$ returns values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we have $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$.

Question 2

Part A

$\theta = 0.9273$ radians

Given:

- Ladder length = 10 feet
- Distance from wall = 6 feet

We can set up a right triangle where:

- The horizontal distance from the wall is the adjacent side = 6 feet
- The ladder length is the hypotenuse = 10 feet
- The angle θ is the angle between the ladder and the ground

Using the cosine relationship:

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{6}{10} = 0.6$$
$$\theta = \cos^{-1}(0.6) \approx 0.9273 \text{ radians}$$

Question 3

Part A

Domain: $\left[\frac{1}{3}, 1\right]$

For $f(x) = \cos^{-1}(3x - 2)$:
The domain of $\cos^{-1}(u)$ is $-1 \leq u \leq 1$. So we need:

$$-1 \leq 3x - 2 \leq 1$$
$$-1 + 2 \leq 3x - 2 + 2 \leq 1 + 2$$
$$1 \leq 3x \leq 3$$
$$\frac{1}{3} \leq x \leq 1$$

Therefore, the domain is $\left[\frac{1}{3}, 1\right]$.

Part B

Range: $[0, \pi]$

The range of $\cos^{-1}(u)$ is $[0, \pi]$ for any input u in $[-1, 1]$.

Since $3x - 2$ takes values from -1 to 1 as x varies in the domain $\left[\frac{1}{3}, 1\right]$, the range of $f(x) = \cos^{-1}(3x - 2)$ is also $[0, \pi]$.

Question 4

Part A

$f^{-1}(x) = 3\sin^{-1}\left(\frac{x}{4}\right)$

Given $f(x) = 4\sin\left(\frac{x}{3}\right)$ for $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$
To find the inverse function:

$$\begin{aligned} y &= 4\sin\left(\frac{x}{3}\right) \\ \frac{y}{4} &= \sin\left(\frac{x}{3}\right) \\ \sin^{-1}\left(\frac{y}{4}\right) &= \frac{x}{3} \\ 3\sin^{-1}\left(\frac{y}{4}\right) &= x \end{aligned}$$

Switching variables, we get $f^{-1}(x) = 3\sin^{-1}\left(\frac{x}{4}\right)$

Part B

Domain of $f^{-1}(x)$: $[-4, 4]$

The domain of $f^{-1}(x)$ is the range of $f(x)$.
For $f(x) = 4\sin\left(\frac{x}{3}\right)$:

- Amplitude = 4
- Range = $[-4, 4]$

Therefore, the domain of $f^{-1}(x)$ is $[-4, 4]$.

Question 5

Part A

$h^{-1}(x) = 4\tan^{-1}\left(\frac{x}{2}\right)$

Given $h(x) = 2\tan\left(\frac{x}{4}\right)$ for $-2\pi < x < 2\pi$
To find the inverse function:

$$\begin{aligned} y &= 2\tan\left(\frac{x}{4}\right) \\ \frac{y}{2} &= \tan\left(\frac{x}{4}\right) \\ \tan^{-1}\left(\frac{y}{2}\right) &= \frac{x}{4} \\ 4\tan^{-1}\left(\frac{y}{2}\right) &= x \end{aligned}$$

Switching variables, we get $h^{-1}(x) = 4\tan^{-1}\left(\frac{x}{2}\right)$

Part B

Domain of $h^{-1}(x)$: $\mathbb{R}$ (all real numbers)

The domain of $h^{-1}(x)$ is the range of $h(x)$.
For $h(x) = 2\tan\left(\frac{x}{4}\right)$ where $-2\pi < x < 2\pi$:

- $\frac{x}{4}$ ranges from $\frac{-2\pi}{4} = \frac{-\pi}{2}$ to $\frac{2\pi}{4} = \frac{\pi}{2}$
- This range covers exactly one period of the function
- The range of $\tan(\theta)$ is $(-\infty, \infty)$
- Multiplying by 2 still gives a range of $(-\infty, \infty)$

Therefore, the domain of $h^{-1}(x)$ is $\mathbb{R}$ (all real numbers).

Question 6

Part A

Domain of $f(x) = \sin(\sin^{-1}(x))$: $[-1, 1]$

The domain of $f(x) = \sin(\sin^{-1}(x))$ is determined by the domain of the inner function $\sin^{-1}(x)$.
Since $\sin^{-1}(x)$ has domain $[-1, 1]$, the domain of $f(x) = \sin(\sin^{-1}(x))$ is also $[-1, 1]$.

Part B

Range of $f(x) = \sin(\sin^{-1}(x))$: $[-1, 1]$

For $f(x) = \sin(\sin^{-1}(x))$, we can simplify:

$$\begin{aligned} f(x) &= \sin(\sin^{-1}(x)) \\ &= x \end{aligned}$$

Since x is in $[-1, 1]$, the range of $f(x) = x$ on this domain is also $[-1, 1]$.

Part C

Domain of $f(x) = \sin^{-1}(\sin(x))$: $\mathbb{R}$ (all real numbers)

The domain of $f(x) = \sin^{-1}(\sin(x))$ is determined by when $\sin(x)$ produces values in the domain of $\sin^{-1}$, which is $[-1, 1]$.
Since $\sin(x)$ always produces values in $[-1, 1]$ for any real input x, the composition is defined for all real numbers.
Therefore, the domain is $\mathbb{R}$ (all real numbers).

Part D

Range of $f(x) = \sin^{-1}(\sin(x))$: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

For $f(x) = \sin^{-1}(\sin(x))$, the range is determined by the range of $\sin^{-1}$, which is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
However, $\sin^{-1}(\sin(x))$ is not always equal to x. Instead:

- If $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\sin^{-1}(\sin(x)) = x$
- If x is outside this interval, $\sin^{-1}(\sin(x))$ maps x to its equivalent angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore, the range of $f(x) = \sin^{-1}(\sin(x))$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Part E

Parts B and C are different because the compositions $\sin(\sin^{-1}(x))$ and $\sin^{-1}(\sin(x))$ have different domains and involve different restrictions on their input values.

Parts B and C differ because:

1. For $\sin(\sin^{-1}(x))$: - The domain is restricted to $[-1, 1]$ due to $\sin^{-1}$ - This composition simplifies to $\sin(\sin^{-1}(x)) = x$ for all x in the domain - The function is the identity function on $[-1, 1]$

2. For $\sin^{-1}(\sin(x))$: - The domain is all real numbers - This composition only simplifies to x on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ - For values outside $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the function maps them to their equivalent angle in this interval

The function $\sin^{-1}(x)$ is not the inverse of $\sin(x)$! It is the inverse of $\sin(x)$ restricted to the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Question 7

Part A

$\frac{1}{\cos(\theta)} = \sec(\theta)$

The reciprocal of cosine is the secant function: $\sec(\theta) = \frac{1}{\cos(\theta)}$

Part B

$\frac{1}{\sin(\theta)} = \csc(\theta)$

The reciprocal of sine is the cosecant function: $\csc(\theta) = \frac{1}{\sin(\theta)}$

Part C

$\frac{1}{\tan(\theta)} = \cot(\theta)$

The reciprocal of tangent is the cotangent function: $\cot(\theta) = \frac{1}{\tan(\theta)}$

Question 8

Part A

$\sec^2(x) - \tan^2(x) = 1$

To simplify $\sec^2(x) - \tan^2(x)$:

$$\begin{aligned}
\sec^2(x) - \tan^2(x) &= \frac{1}{\cos^2(x)} - \frac{\sin^2(x)}{\cos^2(x)} \\
&= \frac{1 - \sin^2(x)}{\cos^2(x)} \\
&= \frac{\cos^2(x)}{\cos^2(x)} \\
&= 1
\end{aligned}$$

Remember that $\cos^2(x) + \sin^2(x) = 1$. This is the Pythagorean identity.

Question 9

Part A

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

We need to solve $\csc(x) = 2$ for $x \in [0, 2\pi]$.
Since $\csc(x) = \frac{1}{\sin(x)}$, we have:

$$\frac{1}{\sin(x)} = 2$$
$$\sin(x) = \frac{1}{2}$$

In the interval $[0, 2\pi]$, $\sin(x) = \frac{1}{2}$ occurs at:

$$x = \frac{\pi}{6} \text{ (first quadrant)}$$
$$x = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \text{ (second quadrant)}$$

Therefore, $\csc(x) = 2$ when $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

Question 10

Part A

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

We need to solve $\sec(x) = -2$ for $x \in [0, 2\pi]$.
Since $\sec(x) = \frac{1}{\cos(x)}$, we have:

$$\frac{1}{\cos(x)} = -2$$
$$\cos(x) = -\frac{1}{2}$$

In the interval $[0, 2\pi]$, $\cos(x) = -\frac{1}{2}$ occurs at:

$$x = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \text{ (second quadrant)}$$
$$x = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3} \text{ (third quadrant)}$$

Therefore, $\sec(x) = -2$ when $x = \frac{2\pi}{3}, \frac{4\pi}{3}$.

Question 11

Part A

$\sec(\cos^{-1}(1/3)) = 3$

Let $\theta = \cos^{-1}(1/3)$. Then $\cos(\theta) = 1/3$.
Now we can find $\sec(\theta)$ directly:

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{1/3} = 3$$

Therefore, $\sec(\cos^{-1}(1/3)) = 3$.

Question 12

Part A

Vertical asymptotes occur at $x = 3\pi, 4\pi, 5\pi$

The function $f(x) = \csc(x)$ has vertical asymptotes where $\sin(x) = 0$, which occurs at $x = n\pi$ for all integers n.
In the interval $[3\pi, 5\pi]$, the values of $n\pi$ that fall within this range are:

- 3π (when $n = 3$)
- 4π (when $n = 4$)
- 5π (when $n = 5$)

Therefore, the vertical asymptotes of $f(x) = \csc(x)$ in $[3\pi, 5\pi]$ occur at $x = 3\pi, 4\pi, 5\pi$.

Question 13

Part A

Vertical asymptotes occur at $x = 12, 15$

The function $f(x) = 2\csc(\frac{\pi}{3}x)$ has vertical asymptotes where $\sin(\frac{\pi}{3}x) = 0$.
This occurs when $\frac{\pi}{3}x = n\pi$ for integers n, which simplifies to $x = 3n$ for integers n.
In the interval $[3\pi, 5\pi]$:

- $3\pi \approx 9.42$ and $5\pi \approx 15.71$
- We need values of $3n$ from approximately 9.42 to 15.71
- This gives us $n = 4, 5$
- So $x = 12, 15$

Therefore, the vertical asymptotes of $f(x) = 2\csc(\frac{\pi}{3}x)$ in $[3\pi, 5\pi]$ occur at $x = 12, 15$.

Question 14

Part A

Domain: $\{x \mid x \neq \frac{7\pi}{4} + 3n\pi \text{ for all integers } n\}$

The function $f(x) = \sec(\frac{1}{3}(x - \frac{\pi}{4}))$ is undefined when $\cos(\frac{1}{3}(x - \frac{\pi}{4})) = 0$.
This occurs when $\frac{1}{3}(x - \frac{\pi}{4}) = \frac{\pi}{2} + n\pi$ for integers n.
Solving for x:

$$\frac{1}{3}(x - \frac{\pi}{4}) = \frac{\pi}{2} + n\pi$$
$$x - \frac{\pi}{4} = \frac{3\pi}{2} + 3n\pi$$
$$x = \frac{\pi}{4} + \frac{3\pi}{2} + 3n\pi$$

Therefore, the domain is all real numbers except $x = \frac{7\pi}{4} + 3n\pi$ for all integers n.

Question 15

Part A

Domain: $\{x \mid x \neq \frac{\pi}{4} + 3n\pi \text{ for all integers } n\}$

The function $f(x) = \csc(\frac{1}{3}(x - \frac{\pi}{4}))$ is undefined when $\sin(\frac{1}{3}(x - \frac{\pi}{4})) = 0$.
This occurs when $\frac{1}{3}(x - \frac{\pi}{4}) = n\pi$ for integers n.
Solving for x:

$$\frac{1}{3}(x - \frac{\pi}{4}) = n\pi$$
$$x - \frac{\pi}{4} = 3n\pi$$
$$x = \frac{\pi}{4} + 3n\pi$$

Therefore, the domain is all real numbers except $x = \frac{\pi}{4} + 3n\pi$ for all integers n.

Question 16

Part A

Domain: $\{x \mid x \neq \frac{\pi}{4} + 3n\pi \text{ for all integers } n\}$

The function $f(x) = \cot(\frac{1}{3}(x - \frac{\pi}{4}))$ is undefined when $\sin(\frac{1}{3}(x - \frac{\pi}{4})) = 0$.
This occurs when $\frac{1}{3}(x - \frac{\pi}{4}) = n\pi$ for integers n.
Solving for x:

$$\frac{1}{3}(x - \frac{\pi}{4}) = n\pi$$
$$x - \frac{\pi}{4} = 3n\pi$$
$$x = \frac{\pi}{4} + 3n\pi$$

Therefore, the domain is all real numbers except $x = \frac{\pi}{4} + 3n\pi$ for all integers n.

Question 17

Part A

Range: $(-\infty, -2.5] \cup [-1.5, \infty)$

For $f(x) = \frac{1}{2}\csc(\pi(x+10)) - 2$:

- The range of $\csc(\theta)$ is $(-\infty, -1] \cup [1, \infty)$
- Multiplying by $\frac{1}{2}$ gives a range of $(-\infty, -0.5] \cup [0.5, \infty)$
- Subtracting 2 shifts this to $(-\infty, -2.5] \cup [-1.5, \infty)$

Therefore, the range of $f(x) = \frac{1}{2}\csc(\pi(x+10)) - 2$ is $(-\infty, -2.5] \cup [-1.5, \infty)$.

Question 18

Graph it in Desmos.