Spring 2025 Laird Homework 4 Solutions

Question 1

$f(x) = 3\sin\left(\frac{2\pi}{5}(x-2)\right) + 10$

Problem

Write a sinusoidal function with a period of 5, an amplitude of 3, a midline of $y = 10$, that passes through $(2, 10)$.

Solution

For a sinusoidal function in the form $f(x) = a\sin(b(x-c)) + d$:

- Amplitude $a = 3$
- Period $= 5$, so $b = \frac{2\pi}{5}$
- Midline $d = 10$

Since the function passes through $(2, 10)$ and 10 is the midline, this means $\sin(b(2-c)) = 0$. When sine equals 0, its input must be a multiple of π.

So $\frac{2\pi}{5}(2-c) = n\pi$, where n is an integer.

Solving for c:

$$2 - c = \frac{5n\pi}{2\pi} = \frac{5n}{2}$$
$$c = 2 - \frac{5n}{2}$$

The simplest solution is when $n = 0$, giving $c = 2$.

Therefore:

$$f(x) = 3\sin\left(\frac{2\pi}{5}(x-2)\right) + 10$$

Question 2

Part A

Daylight minutes are *increasing* at a *decreasing rate* on day 150.

We need to analyze the behavior of the function $D(t) = 160\cos\left(\frac{2\pi}{365}(t-172)\right) + 729$ at $t = 150$.

First, let's identify key features of this cosine function:

- The period is 365 days (one full year)
- The function reaches its maximum value when $\cos\left(\frac{2\pi}{365}(t-172)\right) = 1$, which occurs when $t = 172$
- The function reaches its minimum value when $\cos\left(\frac{2\pi}{365}(t-172)\right) = -1$, which occurs at $t = 172 + \frac{365}{2} = 354.5$

To determine if daylight is increasing or decreasing at $t = 150$, we need to determine where day 150 falls in the cosine cycle:

- Day 150 is 22 days before day 172 (the maximum)
- Since cosine increases from $t = 354.5$ to $t = 172$ (considering the cyclic nature), day 150 falls in this increasing region
- Therefore, daylight minutes are increasing on day 150

To determine if the rate of increase is itself increasing or decreasing:

- The cosine function's rate of change varies throughout its cycle
- It changes most rapidly at the midpoints between extrema (days 263.25 and 445.75/80.75)
- It changes least rapidly near the extrema (days 172 and 354.5)
- Since day 150 is approaching the maximum at day 172, the rate of increase is slowing down
- Therefore, daylight minutes are increasing at a decreasing rate on day 150

We can also verify this graphically: as we approach the peak of a cosine curve, the slope is positive but gradually becoming less steep (approaching zero at the maximum).

Part B

Day 172 (June 21) gets the most daylight.

The maximum daylight occurs when $\cos\left(\frac{2\pi}{365}(t-172)\right) = 1$, which happens when $\frac{2\pi}{365}(t-172) = 0$.

Solving:

$$t - 172 = 0$$
$$\Rightarrow t = 172$$

Therefore, day 172 (June 21) gets the most daylight.

Question 3

Part A

$f(x) = \sin(x + c)$ is an even function when $c = \frac{\pi}{2} + n\pi$ for any integer n.

A function is even if $f(-x) = f(x)$ for all values of x.
For $f(x) = \sin(x + c)$, we have:

$$f(-x) = \sin(-x + c)$$

For f to be even, we need $f(-x) = f(x)$, so:

$$\sin(-x + c) = \sin(x + c)$$

Using the identity $\sin(-\theta) = -\sin(\theta)$, we can rewrite the left side:

$$-\sin(x - c) = \sin(x + c)$$

This equation holds when $\sin(x + c) = -\sin(x - c)$ for all x.
When $c = \frac{\pi}{2}$, let's check:

$$\sin(x + \frac{\pi}{2}) = -\sin(x - \frac{\pi}{2})$$

Using the identities $\sin(x + \frac{\pi}{2}) = \cos(x)$ and $\sin(x - \frac{\pi}{2}) = -\cos(x)$:

$$\cos(x) = -(-\cos(x))$$
$$\cos(x) = \cos(x)$$

This is true for all x. The sine function with a phase shift of $\frac{\pi}{2}$ becomes $\cos(x)$, which is an even function.
Similarly, when $c = \frac{3\pi}{2}, \frac{5\pi}{2}, ...,$ we get $-\cos(x)$, which is also even.
Therefore, $f(x) = \sin(x + c)$ is an even function when $c = \frac{\pi}{2} + n\pi$ for any integer n.

Part B

$f(x) = \sin(x + c)$ is an odd function when $c = n\pi$ for any integer n.

A function is odd if $f(-x) = -f(x)$ for all values of x.
For $f(x) = \sin(x + c)$:

$$f(-x) = \sin(-x + c)$$
$$-f(x) = -\sin(x + c)$$

For f to be odd, we need $\sin(-x + c) = -\sin(x + c)$ for all x.
Using $\sin(-\theta) = -\sin(\theta)$:

$$\sin(-x + c) = -\sin(x - c)$$

So we need $-\sin(x - c) = -\sin(x + c)$, which simplifies to:

$$\sin(x - c) = \sin(x + c)$$

This is true when $x - c$ and $x + c$ differ by a complete period (2π) or when they are supplementary angles.
The simplest case is when $c = 0, \pi, 2\pi$, etc. In other words, $c = n\pi$ where n is any integer.
For example:

- When $c = 0$, $f(x) = \sin(x)$, which is naturally odd
- When $c = \pi$, $f(x) = \sin(x + \pi) = -\sin(x)$, which is also odd

Therefore, $f(x) = \sin(x + c)$ is an odd function when $c = n\pi$, where n is any integer.

Question 4

$h(t) = 2.5 \sin\left(\frac{\pi t}{6}\right)$

Given:

- Maximum height = 2.5 feet
- Minimum height = -2.5 feet
- At $t = 0$, $h = 0$ (dolphin at surface)
- It takes 3 seconds to reach maximum height from surface

For $h(t) = a \sin(bt)$:

- Amplitude $a = 2.5$
- At $t = 0$, $h(0) = 0$
- At $t = 3$, $h(3) = 2.5 = 2.5 \sin(3b)$, so $\sin(3b) = 1$
- This means $3b = \frac{\pi}{2} + 2n\pi$
- The simplest solution is $b = \frac{\pi}{6}$

Therefore:

$$h(t) = 2.5 \sin\left(\frac{\pi t}{6}\right)$$

We can verify:

- $h(0) = 0$ (surface)
- $h(3) = 2.5$ (maximum height)
- $h(6) = 0$ (surface again)
- $h(9) = -2.5$ (minimum height)
- $h(12) = 0$ (back to surface)

This gives a full period of 12 seconds, which matches our scenario perfectly.

Question 6

$d(t) = 2.5\cos(20\pi t)$

Given:

- Initial position is at the top: $d(0) = 2.5$ cm
- Period is 0.1 seconds
- The height varies from -2.5 cm to 2.5 cm

For a cosine function in the form $d(t) = a\cos(bt) + c$:

- Amplitude $a = \frac{2.5-(-2.5)}{2} = 2.5$ cm
- Period = 0.1 seconds, so $b = \frac{2\pi}{0.1} = 20\pi$
- Vertical shift $c = \frac{2.5+2.5}{2} = 2.5$ cm (midline)

Since the needle starts at the top position ($t = 0$, $d = 2.5$):

$$\begin{aligned} d(0) &= 2.5\cos(0) + 2.5 \\ &= 2.5 \cdot 1 + 2.5 \\ &= 5\checkmark \end{aligned}$$

Therefore:

$$d(t) = 2.5\cos(20\pi t) + 2.5$$

We can verify:

- $d(0) = 5$ (top position)
- $d(0.05) = 2.5\cos(20\pi \cdot 0.05) + 2.5 = 2.5\cos(\pi) + 2.5 = 2.5 \cdot (-1) + 2.5 = 0$ (middle position)
- $d(0.1) = 2.5\cos(20\pi \cdot 0.1) + 2.5 = 2.5\cos(2\pi) + 2.5 = 2.5 \cdot 1 + 2.5 = 5$ (bottom position)
- $d(0.15) = 2.5\cos(20\pi \cdot 0.15) + 2.5 = 2.5\cos(3\pi) + 2.5 = 2.5 \cdot (-1) + 2.5 = 0$ (middle position)

This confirms our model accurately represents the needle's motion.

Question 7

The local minima of $h(x)$ occur at $x = \frac{\pi}{4} + n\pi$, where n is any integer.

Starting with $f(x) = \cos(x)$, we apply the transformations:

- Horizontal dilation by factor $\frac{1}{2}$ means the function completes its cycle twice as fast, giving $g(x) = \cos(2x)$.

- A phase shift of $-\frac{\pi}{4}$ gives:

$$\begin{aligned} h(x) &= g\left(x - \left(-\frac{\pi}{4}\right)\right) \\ &= g\left(x + \frac{\pi}{4}\right) \\ &= \cos\left(2\left(x + \frac{\pi}{4}\right)\right) \\ &= \cos\left(2x + \frac{\pi}{2}\right) \end{aligned}$$

Using the identity $\cos\left(A + \frac{\pi}{2}\right) = -\sin(A)$, we can rewrite:

$$h(x) = \cos\left(2x + \frac{\pi}{2}\right) = -\sin(2x)$$

To find local minima of $h(x) = -\sin(2x)$, we need to determine when this function reaches its lowest points.

Since $\sin(\theta)$ has a maximum value of 1, the function $-\sin(\theta)$ has a minimum value of -1.

Therefore, $h(x) = -\sin(2x)$ reaches its minimum value when $\sin(2x) = 1$.

$\sin(\theta) = 1$ occurs when $\theta = \frac{\pi}{2} + 2n\pi$ for any integer n.

So we need:

$$\begin{aligned} 2x &= \frac{\pi}{2} + 2n\pi \\ x &= \frac{\pi}{4} + n\pi \end{aligned}$$

Therefore, the local minima of $h(x)$ occur at $x = \frac{\pi}{4} + n\pi$, where n is any integer.

We can verify these are minima:

- At $x = \frac{\pi}{4}$: $h\left(\frac{\pi}{4}\right) = -\sin\left(2 \cdot \frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$
- At $x = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$: $h\left(\frac{5\pi}{4}\right) = -\sin\left(2 \cdot \frac{5\pi}{4}\right) = -\sin\left(\frac{10\pi}{4}\right) = -\sin\left(\frac{\pi}{2} + 2\pi\right) = -1$

Question 9

The function $f(x) = 2\sin(4x)$ completes 63 full periods in the interval $0 < x < 100$.

We need to determine how many complete periods the function $f(x) = 2\sin(4x)$ has in the interval $0 < x < 100$.

First, let's find the period of this function:

For a function in the form $\sin(bx)$, the period is $\frac{2\pi}{b}$.

In our case, with $f(x) = 2\sin(4x)$:

- The coefficient 2 is the amplitude, which doesn't affect the period.
- $b = 4$, so the period is $\frac{2\pi}{4} = \frac{\pi}{2}$.

Now, to find how many complete periods fit in the interval $0 < x < 100$:

$$\begin{aligned}
\text{Number of periods} &= \frac{\text{Length of interval}}{\text{Period}} \\
&= \frac{100 - 0}{\pi/2} \\
&= \frac{100}{\pi/2} \\
&= \frac{200}{\pi} \\
&\approx 63.6620
\end{aligned}$$

Since we're looking for complete periods, we take the integer part: 63 complete periods.
We can verify this:

- 63 periods cover a distance of $63 \cdot \frac{\pi}{2} = \frac{63\pi}{2} \approx 98.9602$ units
- 64 periods would cover $64 \cdot \frac{\pi}{2} = 32\pi \approx 100.5310$ units, which exceeds our interval

Therefore, the function $f(x) = 2\sin(4x)$ completes exactly 63 periods in the interval $0 < x < 100$.