Solutions to Quiz 3

Problem 1

A) In Triangle I, we need to find angle α.

In a triangle, the sum of all angles equals π radians. We know two angles: $\frac{\pi}{12}$ and $\frac{\pi}{2}$ (right angle). Therefore:

$$\begin{aligned} \alpha &= \pi - \frac{\pi}{12} - \frac{\pi}{2} && (1) \\ &= \pi - \frac{\pi}{12} - \frac{6\pi}{12} && (2) \\ &= \pi - \frac{7\pi}{12} && (3) \\ &= \frac{12\pi - 7\pi}{12} && (4) \\ &= \frac{5\pi}{12} \text{ radians} && (5) \end{aligned}$$

B) In Triangle II, we need to find side x.

This is a right triangle with hypotenuse 15 and one leg 12. We can use the Pythagorean theorem:

$$\begin{aligned} 12^2 + x^2 &= 15^2 && (6) \\ 144 + x^2 &= 225 && (7) \\ x^2 &= 81 && (8) \\ x &= 9 \text{ units} && (9) \end{aligned}$$

C) In Triangle III, we need to find side y.

This is a right triangle with hypotenuse 10 and angles $\frac{\pi}{4}$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$. We'll use the Pythagorean theorem and the fact that the legs are equal (since angles are equal).

Let y be the length of both legs. By the Pythagorean theorem:

$$\begin{aligned} y^2 + y^2 &= 10^2 && (10) \\ 2y^2 &= 100 && (11) \\ y^2 &= 50 && (12) \\ y &= \sqrt{50} = 5\sqrt{2} \text{ units} && (13) \end{aligned}$$

Problem 2

A) Finding the height of an equilateral triangle with sides 6 meters.

Let's split the equilateral triangle into two right triangles. The height h divides the base into two equal parts of length 3.

Using the Pythagorean theorem:

$$\begin{aligned} h^2 + 3^2 &= 6^2 && (14) \\ h^2 + 9 &= 36 && (15) \\ h^2 &= 27 && (16) \\ h &= \sqrt{27} = 3\sqrt{3} \text{ meters} && (17) \end{aligned}$$

B) In the triangle with angles $\frac{\pi}{6}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$ and hypotenuse 7 units, we need to find side a.

Using the sine law:

$$\frac{a}{\sin\left(\frac{\pi}{3}\right)} = \frac{7}{\sin\left(\frac{\pi}{2}\right)} \tag{18}$$

$$\frac{a}{\frac{\sqrt{3}}{2}} = \frac{7}{1} \tag{19}$$

$$a = 7 \cdot \frac{\sqrt{3}}{2} \text{ units} \tag{20}$$

C) Finding the length of side b.
Using the sine law:

$$\frac{b}{\sin\left(\frac{\pi}{6}\right)} = \frac{7}{\sin\left(\frac{\pi}{2}\right)} \tag{21}$$

$$\frac{b}{\frac{1}{2}} = \frac{7}{1} \tag{22}$$

$$b = 7 \cdot \frac{1}{2} = \frac{7}{2} \text{ units} \tag{23}$$

Problem 0 (Bonus)
Given: $\log_{10} \sin x + \log_{10} \cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$
From the first equation:

$$\log_{10}(\sin x \cdot \cos x) = -1 \tag{24}$$

$$\sin x \cdot \cos x = 10^{-1} = \frac{1}{10} \tag{25}$$

From the second equation:

$$\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1) \tag{26}$$

$$\sin x + \cos x = 10^{\frac{1}{2}(\log_{10} n - 1)} = \sqrt{\frac{n}{10}} \tag{27}$$

So we have:

$$\sin x \cdot \cos x = \frac{1}{10} \tag{28}$$

$$\sin x + \cos x = \sqrt{\frac{n}{10}} \tag{29}$$

Using the identity $(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + 2 \sin x \cos x$:

$$\left(\sqrt{\frac{n}{10}}\right)^2 = 1 + 2\left(\frac{1}{10}\right) \tag{30}$$

$$\frac{n}{10} = 1 + \frac{2}{10} = \frac{10+2}{10} = \frac{12}{10} = \frac{6}{5} \tag{31}$$

$$n = \frac{6}{5} \cdot 10 = 12 \tag{32}$$

Problem 3

A) Finding $f(\alpha)$ that expresses the length of side y as a function of angle α.

In the right triangle, we have hypotenuse = 12 and we need to find y in terms of α. Using trigonometric ratios:

$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{12} \tag{33}$$

$$y = 12\sin(\alpha) \tag{34}$$

Therefore: $f(\alpha) = 12\sin(\alpha)$

B) Calculating the exact value of $\sin\left(\frac{\pi}{4}\right)$:

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \tag{35}$$

C) Calculating the exact value of $\sin\left(\frac{2\pi}{3}\right)$:

$$\sin\left(\frac{2\pi}{3}\right) = \sin\left(\pi - \frac{\pi}{3}\right) \tag{36}$$

$$= \sin\left(\frac{\pi}{3}\right) \tag{37}$$

$$= \frac{\sqrt{3}}{2} \tag{38}$$

D) Calculating the exact value of $\sin\left(\frac{7\pi}{6}\right)$:

$$\sin\left(\frac{7\pi}{6}\right) = \sin\left(\pi + \frac{\pi}{6}\right) \tag{39}$$

$$= -\sin\left(\frac{\pi}{6}\right) \tag{40}$$

$$= -\frac{1}{2} \tag{41}$$

E) Solutions to $\sin(\theta) = \frac{1}{2}$ in $[0, 2\pi)$:
$\theta = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$
Also, $\theta = \pi - \arcsin\left(\frac{1}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$
Solutions: $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$

Problem 4

A) Finding the x-coordinate where the terminal ray at $\frac{\pi}{3}$ intersects a circle with radius 4.

The coordinates of a point on a circle with radius r at angle θ are $(r\cos\theta, r\sin\theta)$. Therefore, the x-coordinate is:

$$x = 4\cos\left(\frac{\pi}{3}\right) \tag{42}$$

$$= 4 \cdot \frac{1}{2} \tag{43}$$

$$= 2 \tag{44}$$

B) Finding the x-coordinate where the terminal ray at $\frac{4\pi}{3}$ intersects a circle with radius 400.

$$x = 400\cos\left(\frac{4\pi}{3}\right) \tag{45}$$
$$= 400\cos\left(\pi + \frac{\pi}{3}\right) \tag{46}$$
$$= 400 \cdot (-\cos\left(\frac{\pi}{3}\right)) \tag{47}$$
$$= 400 \cdot \left(-\frac{1}{2}\right) \tag{48}$$
$$= -200 \tag{49}$$

C) Finding the slope of the terminal ray at angle $\frac{3\pi}{4}$.
The slope of a line with angle θ from the positive x-axis is $\tan(\theta)$.

$$\text{slope} = \tan\left(\frac{3\pi}{4}\right) \tag{50}$$
$$= \tan\left(\pi - \frac{\pi}{4}\right) \tag{51}$$
$$= -\tan\left(\frac{\pi}{4}\right) \tag{52}$$
$$= -1 \tag{53}$$

D) Determining the quadrant of angle $\frac{11\pi}{6}$:

$$\frac{11\pi}{6} = \frac{12\pi - \pi}{6} \tag{54}$$
$$= 2\pi - \frac{\pi}{6} \tag{55}$$

Since $2\pi - \frac{\pi}{6}$ is slightly less than 2π, and greater than $\frac{3\pi}{2}$, this angle lies in the fourth quadrant.
E) Determining the quadrant of angle $\frac{11\pi}{4}$:

$$\frac{11\pi}{4} = \frac{8\pi + 3\pi}{4} \tag{56}$$
$$= 2\pi + \frac{3\pi}{4} \tag{57}$$

Since adding 2π completes a full revolution, we're effectively looking at the angle $\frac{3\pi}{4}$, which is in the second quadrant.

Problem 5
A) Calculating the exact value of $\cos\left(\frac{5\pi}{4}\right)$:

$$\cos\left(\frac{5\pi}{4}\right) = \cos\left(\pi + \frac{\pi}{4}\right) \tag{58}$$
$$= -\cos\left(\frac{\pi}{4}\right) \tag{59}$$
$$= -\frac{\sqrt{2}}{2} \tag{60}$$

B) Calculating the exact value of $\tan\left(\frac{5\pi}{4}\right)$:

$$\tan\left(\frac{5\pi}{4}\right) = \tan\left(\pi + \frac{\pi}{4}\right) \tag{61}$$
$$= \tan\left(\frac{\pi}{4}\right) \tag{62}$$
$$= 1 \tag{63}$$

C) Solving $4 = \cos(10x)$ on $[0, 2\pi)$:
Since $|\cos(\theta)| \leq 1$ for all θ, and $4 > 1$, there are no solutions to this equation.
D) Solving $\frac{3}{2} = 1 + \cos(2x)$ on $[2\pi, 4\pi)$:

$$\frac{3}{2} = 1 + \cos(2x) \tag{64}$$
$$\frac{1}{2} = \cos(2x) \tag{65}$$

For $\cos(\theta) = \frac{1}{2}$, the solutions are $\theta = \pm\frac{\pi}{3} + 2n\pi$, where n is an integer.
Therefore:

$$2x = \pm\frac{\pi}{3} + 2n\pi \tag{66}$$
$$x = \pm\frac{\pi}{6} + n\pi \tag{67}$$

For the interval $[2\pi, 4\pi)$, we need to find values of n such that:

$$2\pi \leq \pm\frac{\pi}{6} + n\pi < 4\pi \tag{68}$$

Case 1: $x = \frac{\pi}{6} + n\pi$

$$2\pi \leq \frac{\pi}{6} + n\pi < 4\pi \tag{69}$$
$$2\pi - \frac{\pi}{6} \leq n\pi < 4\pi - \frac{\pi}{6} \tag{70}$$
$$\frac{12\pi - \pi}{6} \leq n\pi < \frac{24\pi - \pi}{6} \tag{71}$$
$$\frac{11\pi}{6} \leq n\pi < \frac{23\pi}{6} \tag{72}$$
$$\frac{11}{6} \leq n < \frac{23}{6} \tag{73}$$

The valid values for n are 2 and 3, giving $x = \frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$ and $x = \frac{\pi}{6} + 3\pi = \frac{19\pi}{6}$.
Case 2: $x = -\frac{\pi}{6} + n\pi$

$$2\pi \leq -\frac{\pi}{6} + n\pi < 4\pi \tag{74}$$
$$2\pi + \frac{\pi}{6} \leq n\pi < 4\pi + \frac{\pi}{6} \tag{75}$$
$$\frac{12\pi + \pi}{6} \leq n\pi < \frac{24\pi + \pi}{6} \tag{76}$$
$$\frac{13\pi}{6} \leq n\pi < \frac{25\pi}{6} \tag{77}$$
$$\frac{13}{6} \leq n < \frac{25}{6} \tag{78}$$

The valid values for n are 3 and 4, giving $x = -\frac{\pi}{6} + 3\pi = \frac{17\pi}{6}$ and $x = -\frac{\pi}{6} + 4\pi = \frac{23\pi}{6}$.

Solutions: $x = 2\pi + \frac{\pi}{6}, 2\pi + \frac{5\pi}{6}, 3\pi + \frac{\pi}{6}, 3\pi + \frac{5\pi}{6}$

E) If $\tan(\theta) = -\frac{3}{4}$, the slope of the terminal ray is $-\frac{3}{4}$.

Consider a right triangle drawn from the origin along the terminal ray, with the opposite side of length 3 units and the adjacent side of length 4 units. This creates a slope of $-\frac{3}{4}$ for the terminal ray.