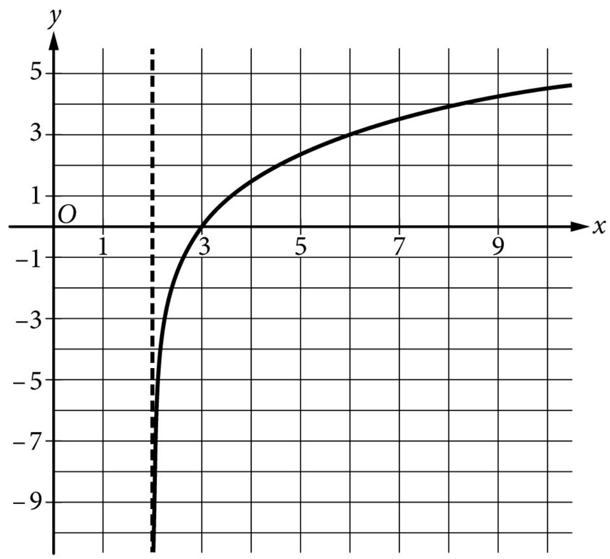
Free-Response Questions Scoring Guidelines

Scoring Guidelines for Question 1: Function Concepts
Part A: Graphing calculator required — 6 points



Graph of f

The figure shows the graph of the increasing function f on its domain of all real numbers $x > 2$. The points $(3, 0)$ and $(6, 3)$ are on the graph of f. The function g is given by $g(x) = \dfrac{9}{(x-3)}$.

Model Solution	Scoring	

A. i. The function h is defined by $h(x) = (g \circ f)(x) = g(f(x))$. Find the value of $h(6)$ as a decimal approximation, or indicate that it is not defined. Show the work that leads to your answer.

ii. Find all real zeros of f, or indicate there are no real zeros.

Model Solution	Scoring	
i. $h(6) = g(f(6)) = g(3)$, which is not defined. (The denominator of the rational function g is 0 when $x = 3$.)	Value is not defined	**1 point** 2.A
ii. From the graph, $x = 3$ is the only real zero.	Zero of f	**1 point** 2.A
	Total for part (A)	**2 points**

B. i. Find all values of x, as decimal approximations, for which $g(x) = -5.8$, or indicate that there are no such values.

ii. Determine the end behavior of g as x decreases without bound. Express your answer using the mathematical notation of a limit.

i. $g(x) = -5.8 \Rightarrow \frac{9}{(x-3)} = -5.8$ $x = 1.448$	Answer	**1 point** **1.A**
ii. As x decreases without bound, the output values approach 0. Therefore, $\lim_{x \to -\infty} g(x) = 0$.	End behavior with limit notation	**1 point** **3.A**
	Total for part (B)	**2 points**

C. i. Is the function f invertible?

ii. Give a reason for your answer in part C (i) based on properties of the function f. Refer to points on the graph of f in your reasoning.

i. f is invertible; f has an inverse function on its domain of $x > 2$.	Answer	**1 point** **1.C**
ii. f is an increasing function. For each point on the graph of f, each output value of f is mapped from a unique input value. There are no repeated $f(x)$ values. A reason that only states "passes the horizontal line test" is not sufficient.	Reason	**1 point** **3.C**
	Total for part (C)	**2 points**
	Total for Question 1	**6 points**

Scoring Guidelines for Question 2: Modeling a Non-Periodic Context
Part A: Graphing calculator required — 6 points

t	0	35	45
Number of Cones	14	57	46

The table gives the number of ice cream cones sold by a food vendor. On the initial day ($t=0$) when the vendor added ice cream cones to the menu, the vendor sold 14 ice cream cones. Thirty-five days later ($t=35$), the vendor sold 57 ice cream cones. Ten days after that ($t=45$), the vendor sold 46 ice cream cones.

The number of ice cream cones sold can be modeled by the quadratic function I given by $I(t)=at^2+bt+c$, where $I(t)$ is the number of ice cream cones sold on day t.

Model Solution	Scoring	

A. i. Use the given data to write three equations that can be used to find the values for constants a, b, and c in the expression for $I(t)$.

ii. Find the values for a, b, and c as decimal approximations.

Model Solution	Scoring	Points
i. Because $I(0)=14$, $I(35)=57$, and $I(45)=46$, three equations to find a, b, and c are $a(0)^2+b(0)+c=14$ $a(35)^2+b(35)+c=57$ $a(45)^2+b(45)+c=46.$	Three equations	**1 point** 1.C
ii. $c=14$ $1225a+35b=43$ $2025a+45b=32$ $-3150a=163$ $a=-\frac{163}{3150}=-0.051746$ $b=3.039683$ $I(t)=-0.052t^2+3.040t+14$ OR $I(t)=-0.051t^2+3.039t+14$	Values of a, b, and c	**1 point** 1.C

Total for part (A) — **2 points**

B. i. Use the given data to find the average rate of change of the number of ice cream cones sold, in cones per day, from $t=35$ to $t=45$ days. Express your answer as a decimal approximation. Show the computations that lead to your answer.

ii. Use the average rate of change found in part B (i) to estimate the number of ice cream cones sold on day $t=40$. Show the work that leads to your answer.

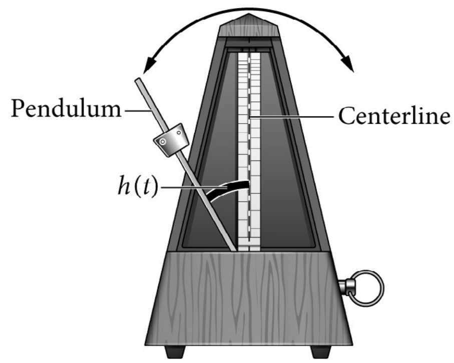
iii. Compare the estimate found in part B (ii) to the value given by the model, $I(40)$. Using characteristics of the average rate of change and characteristics of the quadratic model, explain why the two estimates differ. Your explanation should include a reference to the graph of I.

i. $\dfrac{I(45)-I(35)}{45-35}=\dfrac{(46-57)}{10}=-1.1$ The average rate of change is -1.1 cones per day.	Average rate of change	**1 point** 1.B
ii. The average rate of change is $r=\dfrac{I(45)-I(35)}{45-35}=-1.1$. The secant line between point $(35, I(35))$ and point $(45, I(45))$ is given by $y=y_1+\left(\dfrac{I(45)-I(35)}{45-35}\right)(x-x_1)$, where (x_1, y_1) can be either one of the points. Estimates using the average rate of change are given by $y=I(35)+r(x-35)$ OR $y=I(45)+r(x-45)$. Both of these produce the same estimate. For $x=40$, $y=57+r(40-35)=51.5$. The number of ice cream cones sold on day $t=40$ was approximately 51 or 52.	Estimate using average rate of change	**1 point** 3.B
iii. Based on $I(40)$, the number of ice cream cones sold on day $t=40$ was 52.794, which means 52 or 53 ice cream cones. The estimate using the average rate of change is the y-coordinate of a point on the secant line that passes through $(35, I(35))$ and $(45, I(45))$. Because I is a quadratic function where $a<0$, the graph of I is concave down on its entire domain. The secant line is below the graph of I on the interval $(35, 45)$. Therefore, the estimate found in (ii) using the average rate of change is less than the value of $I(40)$.	Answer with explanation	**1 point** 3.C
	Total for part (B)	**3 points**

C. Explain how the range values of the function I should be limited by the context of the problem.

Based on the context of the number of ice cream cones sold, the range of I consists of nonnegative values. Because I is a quadratic function whose graph is concave down on its domain, there is a maximum value for the number of ice cream cones sold. Based on the model, that maximum is 58.640, which means 58 or 59 ice cream cones. The proposed range is $0 \le I(t) \le 59$.	Answer with reason	**1 point** 3.C
	Total for part (C)	**1 point**
	Total for Question 2	**6 points**

Scoring Guidelines for Question 3: Modeling a Periodic Context
Part B: Graphing calculator not allowed — 6 points



<u>Note</u>: Figure not drawn to scale.

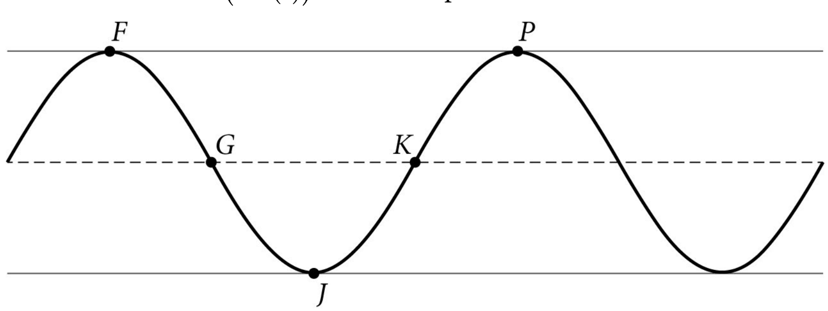
A metronome is a device used to help musicians play music at a particular speed. The metronome has a vertical centerline, as shown in the figure. A pendulum on the metronome swings back and forth as it passes the vertical centerline. When the pendulum is farthest to the left or farthest to the right, the measure of the angle formed by the pendulum and the vertical centerline is 0.5 radian.

At time $t = 0$ seconds, the pendulum is farthest to the left. The pendulum then swings to the right and passes the vertical centerline. At time $t = 2$ seconds, the pendulum is farthest to the right for the first time. Then, the pendulum swings left, passes the vertical centerline, and is farthest to the left again at time $t = 4$ seconds. As the pendulum swings, the measure of the angle formed by the pendulum and the vertical centerline periodically increases and decreases.

The sinusoidal function h models the measure of the angle, in radians, formed by the pendulum and the vertical centerline as a function of time t, in seconds. A negative value of $h(t)$ indicates the pendulum is to the left of the vertical centerline; a positive value of $h(t)$ indicates the pendulum is to the right of the vertical centerline.

Model Solution	Scoring

A. The graph of h and its dashed midline for two full cycles is shown. Five points, F, G, J, K, and P, are labeled on the graph. No scale is indicated, and no axes are presented.
Determine possible coordinates $(t, h(t))$ for the five points: F, G, J, K, and P.



Model Solution	Scoring	
F has coordinates $(2, 0.5)$. G has coordinates $(3, 0)$. J has coordinates $(4, -0.5)$. K has coordinates $(5, 0)$. P has coordinates $(6, 0.5)$. Note: t-coordinates will vary. A correct set of coordinates for one full cycle of h as pictured is acceptable.	$h(t)$-coordinates t-coordinates	**1 point** 2.B **1 point** 2.B

Total for part (A)	**2 points**

B. The function h can be written in the form $h(t) = a\cos(b(t+c)) + d$. Find values of constants a, b, c, and d.

$h(t) = a\cos(b(t+c)) + d$ $a = \frac{1}{2}$ $\frac{2\pi}{b} = 4$, so $b = \frac{2\pi}{4} = \frac{\pi}{2}$ $c = 2$ or $c = -2$ $d = 0$ $h(t) = \frac{1}{2}\cos\left(\frac{\pi}{2}(t+2)\right)$ or $h(t) = \frac{1}{2}\cos\left(\frac{\pi}{2}(t-2)\right)$ OR $a = -\frac{1}{2}$ $\frac{2\pi}{b} = 4$, so $b = \frac{2\pi}{4} = \frac{\pi}{2}$ $c = 0$ $d = 0$ $h(t) = -\frac{1}{2}\cos\left(\frac{\pi}{2}t\right)$ Note: Based on horizontal shifts and reflections, there are other correct forms for $h(t)$.	Vertical transformations: Values of a and d Horizontal transformations: Values of b and c	**1 point** 1.C **1 point** 1.C

Total for part (B)	**2 points**

C. Refer to the graph of h in part A. The t-coordinate of J is t_1, and the t-coordinate of K is t_2.

i. On the interval (t_1, t_2), which of the following is true about h ?

a. h is positive and increasing.

b. h is positive and decreasing.

c. h is negative and increasing.

d. h is negative and decreasing.

ii. On the interval (t_1, t_2), describe the concavity of the graph of h and determine whether the rate of change of h is increasing or decreasing.

i. Choice c.	Function behavior	**1 point** 2.A
ii. The graph of h is concave up on the interval (t_1, t_2), and the rate of change of h is increasing on the interval (t_1, t_2).	Concavity of graph and behavior of rate of change	**1 point** 3.A

Total for part (C)	**2 points**
Total for Question 3	**6 points**

Scoring Guidelines for Question 4: Symbolic Manipulations
Part B: Graphing calculator not allowed — 6 points

Directions:

- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number. Angle measures for trigonometric functions are assumed to be in radians.
- Solutions to equations must be real numbers. Determine the exact value of any expression that can be obtained without a calculator. For example, $\log_2 8$, $\cos\left(\frac{\pi}{2}\right)$, and $\sin^{-1}(1)$ can be evaluated without a calculator.
- Unless otherwise specified, combine terms using algebraic methods and rules for exponents and logarithms, where applicable. For example, $2x+3x$, $5^2 \cdot 5^3$, $\frac{x^5}{x^2}$, and $\ln 3+\ln 5$ should be rewritten in equivalent forms.
- For each part of the question, show the work that leads to your answers.

Model Solution	Scoring	

A. The functions g and h are given by

$$g(x)=15\arcsin x$$
$$h(x)=\log_{10}(1-x)-\log_{10}4.$$

i. Solve $g(x)=5\pi$ for values of x in the domain of g.

ii. Solve $h(x)=1$ for values of x in the domain of h.

Model Solution	Scoring	
i. $g(x)=5\pi$ $15\arcsin x=5\pi$ $\arcsin x=\frac{5\pi}{15}$ $x=\sin\left(\frac{\pi}{3}\right)$ $x=\frac{\sqrt{3}}{2}$	Solution to $g(x)=5\pi$	**1 point** 1.A
ii. $h(x)=1$ $\log_{10}(1-x)-\log_{10}4=1$ $\log_{10}\left(\frac{1-x}{4}\right)=1$ $\left(\frac{1-x}{4}\right)=10$ $1-x=40$ $x=-39$	Solution to $h(x)=1$	**1 point** 1.A
	Total for part (A)	**2 points**

B. The functions j and k are given by

$$j(x)=\log_2(x+4)-11\log_2(x-2)+\log_2\left(x^3\right)$$
$$k(x)=(\cot x)(\csc x).$$

i. Rewrite $j(x)$ as a single logarithm base 2 without negative exponents in any part of the expression. Your result should be of the form $\log_2$ (expression).

ii. Rewrite $k(x)$ as a fraction involving powers of $\cos x$ and no other trigonometric functions.

i. $j(x)=\log_2(x+4)-11\log_2(x-2)+\log_2\left(x^3\right)$ $j(x)=\log_2(x+4)-\log_2\left((x-2)^{11}\right)+\log_2\left(x^3\right)$ $j(x)=\log_2\left(\frac{(x+4)}{(x-2)^{11}}\right)+\log_2\left(x^3\right)$ $j(x)=\log_2\left(\frac{(x+4)\left(x^3\right)}{(x-2)^{11}}\right),\ x>2$	Expression for $j(x)$	**1 point** 1.B
ii. $k(x)=(\cot x)(\csc x)$ $k(x)=\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin x}\right)$ $k(x)=\frac{\cos x}{\sin^2 x}=\frac{\cos x}{1-\cos^2 x},\ \sin x\neq 0$	Expression for $k(x)$	**1 point** 1.B

Total for part (B) **2 points**

C. The function m is given by

$$m(x)=\left(2^x\right)^2-3\cdot 2^x.$$

Find all input values in the domain of m that yield an output value of 18.

$m(x)=18$ $(2^x)^2-3\cdot 2^x=18$ $(2^x)^2-3\cdot 2^x-18=0$ Let $y=2^x$. $y^2-3y-18=0$ $(y-6)(y+3)=0$ $y=6$ or $y=-3$ $2^x=6$ or $2^x=-3$ There is no real value such that $2^x=-3$. $2^x=6$ $\log_2(2^x)=\log_2 6$ $x=\log_2 6$	Includes $(2^x=6$ or $2^x=-3)$	**1 point** 1.A
	$x=\log_2 6$	**1 point** 1.A
	Total for part (C)	**2 points**
	Total for Question 4	**6 points**